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Self-Improving Properties of John–Nirenberg and Poincaré Inequalities on Spaces of Homogeneous Type

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We give a condition which ensures that if one inequality of Sobolev–Poincaré type is valid then other stronger inequalities of a similar type also hold, including weighted versions. Our main result includes many previously known results as special cases. We carry out the analysis in the context of spaces of homogeneous type, but the main result is new even in the usual Euclidean setting. © 1998 Academic Press

1. INTRODUCTION

The purpose of this paper is to unify and generalize some results that have appeared recently concerning Poincaré inequalities. We are interested in knowing when the existence of one inequality of this type implies that others also hold. This question has been studied recently by several authors, but the approach we will use is different, our key result being an

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endpoint weak type estimate obtained from an appropriate good-λ inequality (cf. [BG]). Our approach also allows us to consider other classical situations such as Lipschitz and B.M.O. spaces.

We have been especially motivated by the works [HaK], whose motivation in turn came from the theory of quasiregular mappings (see [HK1]), by [SCo1,2], [MSC] and [BCLSC] with a more geometrical point of view, and by [BM1,2] concerning Dirichlet forms. This paper also contains many of the results obtained in [FLW], where the sharp Poincaré inequality for Hörmander vector fields is obtained via a representation formula. Poincaré results of this type are also proved in [L], [CDG] and [GN].

Although we shall be working in the very general framework of spaces of homogeneous type, many of our results are new even for the usual Euclidean structure of $\mathbf{R}^n$ endowed with Lebesgue measure. In this case, the results and proofs are generally somewhat simpler (see the Appendix).

We are going to consider two types of L^1 inequalities as starting points. The ultimate goal is to show that these inequalities are self-improving, in the sense that they lead to L^p estimates for $p > 1$. Perhaps the most classical example of the first kind of inequality that we will consider is given in $\mathbf{R}^n$ by assuming that a particular function f satisfies

$$\frac{1}{|B|}\int_B |f(y) - f_B|\, dy \leqslant C\, |B|^\alpha \tag{1}$$

for all balls $B \subset \mathbf{R}^n$, with $f_B = |B|^{-1} \int_B f(y)\, dy$ and a constant C which is independent of B. The class of all such f coincides with the Lipschitz space $\Lambda(\alpha)$ when α is positive and with B.M.O. when $\alpha = 0$. It is well-known that if f satisfies (1), then f also satisfies a similar inequality with the L^1 average replaced by the L^p average, $1 < p < \infty$, with a possibly larger constant C.

The model example for the second kind of inequality that we will consider is the (L^1, L^1) Poincaré inequality in $\mathbf{R}^n$ (see for instance [EG] or [HKM]),

$$\frac{1}{|B|}\int_B |f(y) - f_B|\, dy \leqslant C\,\frac{r(B)}{|B|}\int_B |\nabla f|\, dy, \tag{2}$$

where B is any ball in $\mathbf{R}^n$, f is an arbitrary Lipschitz function, and the constant C is independent of both B and f. Note that this estimate is valid not just for a single function f but for a class of functions. As is well-known (see, for example, [EG]), there is a "sharp" version of (2) in which the L^1 average on the left is replaced by the $L^{n'}$ average, $n' = n/(n-1)$, while the the right side remains unchanged.

Another basic example of this type occurs in the space $(\mathbf{R}^n, \rho, dx)$ where ρ is the metric associated with a collection $X_1, ..., X_m$ of Hörmander vector fields (cf. [FP], [NSW] and [SCa]). Here the starting point is the estimate from [Je],

$$\frac{1}{|B|}\int_B |f(x)-f_B|\,dx \leqslant C\frac{r(B)}{|B|}\int_B |Xf(x)|\,dx, \tag{3}$$

where $|Xf| = \sum |X_i f|$. Now, there is a sharp version of this inequality obtained in [FLW] (see also [CDG]) in which the L^1 average on the left is replaced by the $L^{Q'}$ average, where Q is the "homogeneous" dimension and $1/Q + 1/Q' = 1$. A sharp version of the corresponding inequality in which there is an L^p, $p>1$, average on the right was first proved in [L]. These sharp estimates will be special cases of our results.

We can also apply our results to Buser's version [Bu] of the Poincaré inequality for complete manifolds (M, g) with Ricci curvature *Ric* bounded below by $-a^2 g$ where $a \geqslant 0$. Here the gradient and measure are the Riemannian ones:

$$\frac{1}{|B|}\int_B |f-f_B| \leqslant Ce^{aCr(B)}\frac{r(B)}{|B|}\int_B |\nabla f|. \tag{4}$$

The sharp version can be found in [MSC], but it will follow from Theorem 3.1 below and by giving explicit constants as in [FW]. Indeed, by the Bishop–Gromov comparison theorem ([GHL], Theorem 4.19), we can give an explicit bound for the doubling constant associated with the Riemannian metric and the Riemannian volume (see also [SCo3]). In [MSC], other inequalities on manifolds can be found.

A somewhat less-known result that we can use as a starting point is

$$\frac{1}{|B|}\int_B |f(y)-\pi(y)|\,dy \leqslant C\frac{r(B)^m}{|B|}\int_B |\nabla^m f|\,dy, \tag{5}$$

for some polynomial π depending on f and B of degree at most $m-1$, where m is a positive integer. Here $\nabla^m f = \{D^\sigma f\}_{|\sigma|=m}$ and $|\nabla^m f| = \sum |D^\sigma f|$. Estimates of this type can be found for example in [DS], Theorem 3.4, or [Bo]. By the version of Theorem 3.1 in $\mathbf{R}^n$ and for polynomials, namely Theorem 7.3 (together with Example 2.2), we obtain an estimate similar to (5) but with the L^r average on the left for the optimal value $r = n/(n-m)$, $1 < m < n$.

Motivated by these examples, we will first consider inequalities of the form

$$\frac{1}{\mu(B)}\int_B |f-f_B|\,d\mu \leqslant ca(B), \tag{6}$$

which we will refer to as "function space inequalities," and, second, inequalities of the form

$$\frac{1}{\mu(B)} \int_B |f - f_B| \, d\mu \leqslant cb(B, f), \tag{7}$$

which we will call "Poincaré type inequalities." In either case, μ is a measure and $\mu(B)$ denotes the μ-measure of B. An inequality of the first form is generally assumed to hold for a single function f, while one of the second form is assumed to hold for a class of functions.

The main purpose of this paper is to show that under certain mild discrete conditions of geometric type on the functionals a and b, both inequalities (6) and (7) encode an intrinsic L^r self-improving property.

Throughout this paper, (S, d, μ) will denote a space of homogeneous type with a continuous quasimetric d, so that each ball $B(x, r)$ is open. We will recall the definition and basic properties of such a space in Section 4. We use $\mathscr{B}$ to denote the class of all balls B in S, and K to denote the quasimetric constant for d.

2. FUNCTION SPACE INEQUALITIES

We shall consider general functionals of the form $a \colon \mathscr{B} \to (0, \infty)$. The functional a is not necessarily radial. i.e., a need not be of the form $a(B) = \varphi(r(B))$ where $r(B)$ is the radius of B and $\varphi \colon (0, \infty) \to (0, \infty)$. Our model example is the fractional average

$$a(B) = \frac{r(B)^\alpha}{\mu(B)} \, v(B), \tag{8}$$

where $\alpha \geqslant 0$ and v is a nonnegative function or measure. As we shall see, it is also interesting to assume that v is absolutely continuous with respect to μ, with $dv = g \, d\mu$ for $g \in A_\infty(\mu)$, where $A_\infty(\mu)$ denotes the class of weight functions of C. Fefferman and B. Muckenhoupt, i.e., the class of nonnegative functions g for which there exist constants $c, \varepsilon > 0$ such that for any ball B and any μ-measurable set $E \subset B$,

$$\frac{v(E)}{v(B)} \leqslant c \left(\frac{\mu(E)}{\mu(B)} \right)^\varepsilon.$$

We impose the following discrete condition on the functional a relative to a locally integrable weight function w. The condition may be thought of as one which reflects the geometry of the space. We use the notation $w(B)$ to denote $\int_B w \, d\mu$.

DEFINITION 2.1. Let $1 \leqslant r < \infty$ and let w be a weight. We say that the functional a satisfies the (weighted) D_r condition if there exists a finite constant c such that for each ball B and any family $\{B_i\}$ of pairwise disjoint subballs of B,

$$\sum_i a(B_i)^r\, w(B_i) \leqslant c^r a(B)^r\, w(B). \tag{9}$$

We denote the smallest constant c for which (9) holds by $\|a\|$.

Observe that by Hölder's inequality, the family $\{D_r\}$ is decreasing as r increases, that is, if $r < s$, and $a \in D_s$ then $a \in D_r$: in fact, if $r < s$, $a \in D_s$ and $\{B_i\}$ is a pairwise disjoint collection of balls in B, we have

$$\begin{aligned} \sum_i a(B_i)^r\, w(B_i) &\leqslant \left(\sum_i a(B_i)^s\, w(B_i)\right)^{r/s} \left(\sum_i w(B_i)\right)^{1-r/s} \\ &\leqslant \|a\|^r\, a(B)^r\, w(B). \end{aligned}$$

EXAMPLE 2.2. As an example of (9), let us show that if $w = 1$ (so that $w(B) = \mu(B)$ for all B), then for any measure v the fractional average (8) satisfies the unweighted $D_{d/(d-\alpha)}$ condition, $0 < \alpha < d$, where d is the doubling order of μ, i.e., where

$$\mu(B_2) \leqslant c \left\{\frac{r(B_2)}{r(B_1)}\right\}^d \mu(B_1)$$

whenever B_1 and B_2 are balls with $B_1 \subset B_2$.

In fact, we then have, with $r = d/(d-\alpha)$ (so that $\alpha r = d(r-1)$),

$$a(B_i)^r\, \mu(B_i) = \left\{\frac{r(B_i)^d}{\mu(B_i)}\right\}^{r-1} v(B_i)^r \leqslant \left\{c\,\frac{r(B)^d}{\mu(B)}\right\}^{r-1} v(B_i)^r$$

by doubling when $B_i \subset B$, and consequently, if $\{B_i\}$ is a pairwise disjoint collection of balls in B,

$$\begin{aligned} \sum a(B_i)^r\, \mu(B_i) &\leqslant \left\{c\,\frac{r(B)^d}{\mu(B)}\right\}^{r-1} \sum v(B_i)^r \\ &\leqslant \left\{c\,\frac{r(B)^d}{\mu(B)}\right\}^{r-1} \left[\sum v(B_i)\right]^r \\ &\leqslant C\left\{\frac{r(B)^d}{\mu(B)}\right\}^{r-1} v(B)^r = C a(B)^r\, \mu(B). \end{aligned} \tag{10}$$

We use the notation

$$\|g\|_{L^{r,\infty}(B,w)} = \sup_{\lambda>0} \lambda \left(\frac{w(\{x \in B : |g(x)| > \lambda\})}{w(B)} \right)^{1/r}$$

for the normalized weak L^r norm, where $w(E)$ denotes $\int_E w \, d\mu$ for any measurable set E.

Let us now state the main result of this section.

THEOREM 2.3. *Let B_0 be a ball and let $\eta = 17K^7$. Suppose that the functional a satisfies the weighted D_r condition* (9) *for some $1 \leqslant r < \infty$ and some $w \in A_\infty(\mu)$. Let f be a function on ηB_0 such that for all balls B with $B \subset \eta B_0$,*

$$\frac{1}{\mu(B)} \int_B |f - f_B| \, d\mu \leqslant \|f\|_a \, a(B). \tag{11}$$

Then there exists a constant c independent of f and B_0 such that

$$\|f - f_{B_0}\|_{L^{r,\infty}(B_0, w)} \leqslant c \, \|a\| \, \|f\|_a \, a(\eta B_0). \tag{12}$$

Inequality (12) has been improved recently in [MP] by showing that η can be replaced by $(1+\delta)K$ with $\delta > 0$ arbitrarily small. The proof does not use dyadic sets.

COROLLARY 2.4. *Let $1 < r < \infty$. Then with the same hypotheses as in Theorem* 2.3, *if p satisfies $1 < p < r$, there exists a constant c independent of f and B_0 such that*

$$\left(\frac{1}{w(B_0)} \int_{B_0} |f - f_{B_0}|^p \, w \, d\mu \right)^{1/p} \leqslant c \, \|a\| \, \|f\|_a \, a(\eta B_0). \tag{13}$$

Remark 2.5. It follows immediately that the corollary still holds if we only assume that $a \in D_p$ for each $1 < p < r$.

Remark 2.6. We can show that it is sometimes possible to take $\eta = 1$ in both Theorem 2.3 and Corollary 2.4. For example, in the case of Euclidean space with the usual Euclidean metric and a doubling measure μ, and for cubes instead of balls, we will show this in Theorem 7.2. For more general homogeneous spaces, we can take $\eta = 1$ if the quasimetric d and the functional a satisfy additional conditions. In fact, if $1 < r < \infty$ and B_0 satisfies the Boman chain condition (see, e.g., [C], [FGuW], [FLW]), then we can take $\eta = 1$ in (12) provided we assume that (9) holds for every collection $\{B_i\}$ of subballs of B with bounded overlaps. The proof is based on (12) and is essentially the same as the one given in [C] for strong-type estimates and for more restricted functionals a. At the end of Section 5, we make some additional comments about what changes are needed in the argument given in [C]. It is proved in [FGuW], Theorem 5.4, that d-balls satisfy the Boman chain condition if the quasimetric d has the segment property, i.e., if for each pair x, y of points in S there exists a continuous curve γ connecting x and y such that $d(\gamma(t), \gamma(s)) = |t-s|$. For instance, the metric associated with a family of Lipschitz continuous vector fields has the segment property locally (see [FGuW] and also [GN]). Moreover, any complete Riemannian manifold has the (global) segment property by the Hopf–Rinow theorem (see [GHL], Theorem 2.103 and Corollary 2.105).

Theorem 2.3 and Corollary 2.4 will be proved in Section 5. In fact, there is a more general version of Theorem 2.3, and consequently also a more general version of Corollary 2.4 and Remark 2.5, in which the functional $a(B)$ which appears on the *right* side of the D_r hypothesis (9) is replaced by a different functional $a'(B)$ (but the original a still appears on the *left* side of the D_r condition and also in hypothesis (11)). In this case the factor $a(\eta B_0)$ in the conclusion (12) (and in (13)) is replaced by $a'(\eta B_0)$. More precisely we have:

THEOREM 2.7. *Let B_0 be a ball, $\eta = 17K^7$, $1 \leqslant r < \infty$ and $w \in A_\infty(\mu)$. Suppose that for all balls $B \subset \eta B_0$, a locally integrable function f satisfies* (11) *and the functional a verifies the condition*

$$\sum a(B_i)^r \, w(B_i) \leqslant \|a\|^r \, a'(B)^r \, w(B) \tag{14}$$

whenever $\{B_i\}$ is a collection of pairwise disjoint balls in B, where a' is another functional acting on $\mathscr{B}$. Then

$$\|f - f_{B_0}\|_{L^{r,\infty}(B_0, w)} \leqslant c \, \|a\| \, \|f\|_a \, a'(\eta B_0). \tag{15}$$

EXAMPLE 2.8. An important example of the situation in Theorem 2.7 is the following weak fractional version of (8): let $\lambda > 1$ be a fixed number and define

$$a(B) = \frac{r(B)^\alpha}{\mu(B)} \, v(\lambda B). \tag{16}$$

Let $\alpha < d$, where d is the doubling order of μ. We will show in Section 6 that although a is not in $D_{d/(d-\alpha)}$ (with $w = 1$), it is true that for each $r < d/(d-\alpha)$, a satisfies (14) with $w = 1$; that is, if $r < d/(d-\alpha)$ then

$$\sum a(B_i)^r \, \mu(B_i) \leqslant c a'(B)^r \, \mu(B) \tag{17}$$

whenever $\{B_i\}$ is a collection of pairwise disjoint balls in B, where a' is defined by

$$a'(B) = \frac{r(B)^\alpha}{\mu(B)} \, \nu(\lambda' B)$$

for an appropriate value $\lambda' \geqslant \lambda$. In the case of Euclidean space with the usual Euclidean metric, we can always pick $\lambda' = \lambda$.

In particular, by the analogue for a' of Remark 2.5, it follows that if f satisfies

$$\frac{1}{\mu(B)} \int_B |f - f_B| \, d\mu \leqslant c \, \frac{r(B)^\alpha}{\mu(B)} \, \nu(\lambda B)$$

for all balls B and some fixed $\lambda > 1$ and $\alpha < d$, then there exists $\lambda' \geqslant \lambda$ such that for any p with $1 < p < d/(d - \alpha)$,

$$\left(\frac{1}{\mu(B)} \int_B |f - f_B|^p \, d\mu \right)^{1/p} \leqslant C \, \frac{r(B)^\alpha}{\mu(B)} \, \nu(\lambda' B).$$

There is an L^p version of the functional (16) that is related to the concept of very weak derivative (or upper gradient) introduced by J. Heinonen and P. Koskela in [HK1], [HK2]. In fact, in Section 6 we will derive an analogous estimate for functionals of the form

$$a(B) = r(B)^\alpha \left(\frac{\nu(\lambda B)}{\mu(B)} \right)^{1/p_0}, \qquad \lambda > 1,$$

in case $1 \leqslant p_0 < d/\alpha$, the result being that $a(B)$ satisfies condition (17) for $r < p_0 d/(d - \alpha p_0)$ with

$$a'(B) = r(B)^\alpha \left(\frac{\nu(\lambda' B)}{\mu(B)} \right)^{1/p_0}$$

for an appropriate $\lambda' \geqslant \lambda$. For example, such functionals $a(B)$ include the form

$$a(B) = r(B)^\alpha \left(\frac{1}{\mu(B)} \int_{\lambda B} |Xf|^{p_0} \, d\mu \right)^{1/p_0}$$

by choosing $d\nu = |Xf|^{p_0} \, d\mu$.

Remark 2.9. In Corollary 2.4, we do not know in general how to prove the strong type result (13) with $p = r$. In Section 3, however, we will show that it is possible to obtain that endpoint estimate under somewhat stronger initial assumptions.

Here, we show another case in which it is possible to get the strong endpoint estimate. Let a be the fractional average (8), and assume that μ is doubling of order d and $0<\alpha<d$. By Example 2.2, we know that $a\in D_{d/(d-\alpha)}$ with $w=1$. It can be shown for some particular choices of v that a is not in $D_{d/(d-\alpha)+\varepsilon}$ for any $\varepsilon>0$ (see Example 6.1 below). Thus the class D_r does not generally have a self-improving or openness property of the sort

$$D_r \Rightarrow D_{r+\varepsilon}.$$

If, for some inherent reason, the functional satisfies a better D condition than expected, then by Corollary 2.4, the optimal strong type result does hold. As an example, we consider the fractional average

$$a(B)=r(B)^{\alpha}\left(\frac{v(B)}{\mu(B)}\right)^{1/p_0}, \qquad v\in A_{\infty}(\mu).$$

When $p_0=1$, we know that a satisfies the unweighted $D_{d/(d-\alpha)}$ condition, where d is the doubling order of μ. However, in this case, we will show in Section 6 that $a\in D_{d/(d-\alpha)+\varepsilon}$ where $\varepsilon>0$ depends on the A_∞ constant of v. More generally, if $1\leqslant p_0<d/\alpha$, we will show that $a\in D_{p_0 d/(d-\alpha p_0)+\varepsilon}$ for some $\varepsilon>0$.

Thus we obtain the following result, which for simplicity we state in a global form.

Corollary 2.10. *Let d be the doubling order of μ, $1\leqslant p_0<\infty$, $0<\alpha<d/p_0$ and $v\in A_{\infty}(\mu)$. Let f be a locally integrable function satisfying*

$$\frac{1}{\mu(B)}\int_B |f-f_B|\,d\mu \leqslant Cr(B)^{\alpha}\left(\frac{v(B)}{\mu(B)}\right)^{1/p_0} \tag{18}$$

for all balls B, with C independent of B. Then, for all B,

$$\left(\frac{1}{\mu(B)}\int_B |f-f_B|^r\,d\mu\right)^{1/r} \leqslant cr(B)^{\alpha}\left(\frac{v(\eta B)}{\mu(B)}\right)^{1/p_0}, \tag{19}$$

where $r=p_0 d/(d-\alpha p_0)$, $\eta=17K^7$ and c is a multiple of C which is independent of B.

3. POINCARÉ TYPE INEQUALITIES

We now consider a functional $b(B, f)$ of two variables of the form

$$b\colon \mathscr{B} \times \mathscr{F} \to (0, \infty),$$

where $\mathscr{F}$ is an appropiate set of functions contained in $L^1_{loc}(S)$. In applications, the main examples of $\mathscr{F}$ are the Lipschitz class, or Sobolev classes, although our results are not restricted to these classical spaces.

Typical examples of b are those associated with Poincaré inequalities, namely

$$b(B, f) = b_X(B, f) = \frac{r(B)^\alpha}{\mu(B)} \int_B |Xf| \, dv, \tag{20}$$

where X is a differential operator with $X1 = 0$, i.e., with no zero order term. In particular, in Euclidean space, X could be ∇^m or some other appropriate combination of partial derivatives. In case all the derivatives are of first order, we can then take $\mathscr{F} = \Lambda_1$, the Lipschitz class, since such functions are differentiable almost everywhere by the Rademacher–Stepanov theorem.

The main property we need is a certain "stability" property under truncations. This idea was originally introduced in [LN] and exploited in [SW], [FGaW], [FLW] and [BCLSC].

Given a nonnegative function g, the truncation $\tau_\lambda(g)$ is defined by

$$\tau_\lambda(g) = \min\{g, 2\lambda\} - \min\{g, \lambda\} = \begin{cases} 0 & \text{if } \; g(x) \leqslant \lambda \\ g(x) - \lambda & \text{if } \; \lambda < g(x) \leqslant 2\lambda \\ \lambda & \text{if } \; g(x) > 2\lambda. \end{cases}$$

We shall assume that the class $\mathscr{F}$ has the following properties:

- $f \in \mathscr{F} \Rightarrow f + \lambda, \lambda f \in \mathscr{F}$ for $\lambda \in \mathbf{R}$
- $f \in \mathscr{F} \Rightarrow |f| \in \mathscr{F}$
- $f \in \mathscr{F} \Rightarrow \tau_\lambda(|f|) \in \mathscr{F}$ for $\lambda \geqslant 0$.

We also assume that the following natural relationships between the functional b and $\mathscr{F}$ hold:

- $b(B, f) = b(B, f + \lambda)$ for all $f \in \mathscr{F}$ and $\lambda \in \mathbf{R}$
- $b(B, |f|) \leqslant b(B, f)$ for all $f \in \mathscr{F}$

- There exist $r>1$ and a constant C such that for any nonnegative $f \in \mathscr{F}$, any ball B and any sequence λ_k of the form $\{\lambda_k = 2^k \lambda\}$, $k=1, 2, ..., \lambda>0$, we have

$$\sum_{k=1}^{\infty} b(B, \tau_{\lambda_k}(f))^r \leqslant cb(B, f)^r. \tag{21}$$

For example, observe that for the functional $b = b_X$ defined in (20), we have

$$b(B, \tau_{\lambda_k}(f)) = \frac{r(B)^\alpha}{\mu(B)} \int_B |X(\tau_{\lambda_k}(f))| \, dv = \frac{r(B)^\alpha}{\mu(B)} \int_{\{x \in B : \lambda^k < f(x) \leqslant \lambda^{k+1}\}} |Xf| \, dv,$$

and then (21) readily follows since the domains of integration are disjoint.

We assume that b and $\mathscr{F}$ have all the properties listed above and that b also satisfies the following condition (a weighted D_r condition which is uniform in f for $f \in \mathscr{F}$) for some $r \geqslant 1$ and some $w \in A_\infty(\mu)$:

$$\sum_i b(B_i, f)^r \, w(B_i) \leqslant C^r b(B, f)^r \, w(B), \tag{22}$$

for all $f \in \mathscr{F}$, every ball $B \in \mathscr{B}$ and every family $\{B_i\}$ of pairwise disjoint subballs of B.

THEOREM 3.1. *Let B_0 be a ball, $1 \leqslant r < \infty$ and $\eta = 17K^7$. Suppose that the functional b and the class $\mathscr{F}$ satisfy the conditions above (including (21) and (22)) with exponent r and with $w \in A_\infty(\mu)$ for all $f \in \mathscr{F}$ and all balls $B \subset \eta B_0$. Suppose also that the following initial condition holds for all such f and B,*

$$\frac{1}{\mu(B)} \int_B |f - f_B| \, d\mu \leqslant cb(B, f), \tag{23}$$

with c independent of f and B. Then

$$\left(\frac{1}{w(B_0)} \int_{B_0} |f - f_{B_0}|^r \, w \, d\mu \right)^{1/r} \leqslant Cb(\eta B_0, f) \tag{24}$$

with C independent of B_0 and $f \in \mathscr{F}$.

In fact, as the proof will show, the strong type conclusion (24) for such functionals b follows from the corresponding weak type estimate with the same value of r.

As an application, we will derive the following corollary. We say that a weight v belongs to the class $A_p(v)$ (see [M]), $1 \leqslant p < \infty$, if

$$\left(\frac{1}{v(b)} \int_B v \, dv \right) \left(\frac{1}{v(B)} \int_B v^{-p'/p} \, dv \right)^{p/p'} \leqslant C, \quad 1 < p < \infty,$$

where $1/p + 1/p' = 1$, and

$$\frac{1}{v(B)} \int_B v \, dv \leqslant C \, essinf_B v, \quad p = 1,$$

for all balls B, with C independent of B. It will sometimes be convenient to also use the notation $|B|_{dv}$ for the v-measure $v(B)$ of a ball B.

COROLLARY 3.2. *Let μ and v be doubling measures, $p_0 \geqslant 1$ and X be a differential operator for which*

$$\frac{1}{\mu(B)} \int_B |f - f_B| \, d\mu \leqslant Cr(B) \left(\frac{1}{v(B)} \int_B |Xf|^{p_0} \, dv \right)^{1/p_0} \tag{25}$$

for all balls B and all Lipschitz functions f. Let $1 \leqslant p_0 \leqslant p \leqslant q < \infty$ and let (w, v) be a pair of weights such that $w \in A_\infty(\mu)$, $v \in A_{p/p_0}(v)$, and the following balance condition holds:

$$\frac{r(\tilde{B})}{r(B)} \left(\frac{|\tilde{B}|_{wd\mu}}{|B|_{wd\mu}} \right)^{1/q} \leqslant C \left(\frac{|\tilde{B}|_{vdv}}{|B|_{vdv}} \right)^{1/p} \tag{26}$$

for all balls $\tilde{B}$, B such that $\tilde{B} \subset B$. Then

$$\left(\frac{1}{w(B)} \int_B |f - f_B|^q \, w \, d\mu \right)^{1/q} \leqslant Cr(B) \left(\frac{1}{v(B)} \int_{\eta B} |Xf|^p \, v \, dv \right)^{1/p} \tag{27}$$

with C independent of f and B.

The balance condition (26) was introduced in [ChW].

We point out that for $A_\infty(d\mu)$ weights w, we recover the main results of [FLW] without making use of the representation formula obtained in that paper. For instance, starting with (25) with $p_0 = 1$ in the unweighted situation, we have that if X is a collection of smooth vector fields satisfying the Hörmander condition on, say, a ball λB_0, $\lambda > 1$, then there is a constant c such that

$$\left(\frac{1}{|B|} \int_B |f - f_B|^Q \, dx \right)^{1/Q} \leqslant c \, \frac{r(B)}{|B|} \int_B |Xf(x)| \, dx \tag{28}$$

for each ball $B \subset B_0$, where Q is the homogeneous dimension of the vector fields and $|B|$ is the Lebesgue measure of B. Recall that the balls here are not Euclidean balls, but rather they are defined in terms of the metric associated with the vector fields. There are corresponding results for the two weight case. We initially obtain these results with the larger ball ηB on the right side, but we may then take $\eta = 1$ by using the Boman chain condition as in [FGuW] and [FLW]. Moreover, the conclusion of Theorem 3.1 is valid with $\eta = 1$ if we assume im addition to the hypotheses there that B_0 satisfies the Boman chain condition and the functional b satisfies (22) whenever $\{B_i\}$ is a collection of subballs of B with bounded overlaps.

To explain the balance condition, which is essentially necessary in any case, observe in particular that when $p_0 = p$ and $w = v = 1$ in the usual Euclidean case for $\mathbf{R}^n$, the balance condition implies that any q value with $q \leqslant p^* = pn/(n-p)$ will verify (27). Therefore, the sharp value occurs for the Sobolev exponent $q = p^*$.

Remark 3.3. As was the case for Theorem 2.3, there is also a more general version of Theorem 3.1 in which the functional $b(B, f)$ on the right side of (22) is replaced by a different functional $b'(B, f)$, while the original b remains on the left side. In this case, hypothesis (23) remains unchanged, but in the conclusion (24), the new functional $b'(\eta B_0, f)$ appears on the right side. It is also necessary to then assume (21) with b' on both sides, and to assume that $b'(B, |f|) \leqslant b'(B, f)$ for all B and all $f \in \mathscr{F}$. Only minor changes in the proof of Theorem 3.1 are needed in order to obtain this more general version: see the comments in Section 5 after the proof of Theorem 3.1.

4. DEFINITIONS

In this section, we briefly recall some basic definitions and facts about spaces of homogeneous type that we will need.

A quasimetric d on a set S is a function $d: S \times S \to [0, \infty)$ satisfying

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$ for all x, y

(iii) There exists a finite constant $K \geqslant 1$ such that

$$d(x, y) \leqslant K(d(x, z) + d(z, y))$$

for all x, y, z.

Given $x \in S$ and $r > 0$, we let $B(x, r) = \{y \in S : d(x, y) < r\}$ and refer to $B(x, r)$ as the ball with center x and radius r. If μ is a measure and E is a measurable set, we use $\mu(E)$ to denote the μ-measure of E. We sometimes use the notation $|E|_\mu$ instead of $\mu(E)$.

DEFINITION 4.1. A space of homogeneous type (S, d, μ) is a set S together with a quasimetric d and a nonnegative Borel measure μ on S such that $\mu(B(x, r))$ is finite for all $x \in S$ and $r > 0$, and the doubling condition

$$0 < \mu(B(x, 2r)) \leqslant C\mu(B(x, r)), \tag{29}$$

holds for all $x \in S$ and $r > 0$.

The doubling assumption (29) is global in nature, i.e., it is assumed to hold for all $x \in S$ and all $r > 0$. In many important cases, however, doubling is a local property, limited to points x in compact sets and to small values of r. In such cases, our main results then also only hold locally.

The balls $B(x, r)$, $r > 0$, are not necessarily open, but by a theorem of Macias and Segovia [MS], one can always find a continuous quasimetric d' which is equivalent to d (i.e., there are constants c_1 and c_2 such that $c_1 d'(x, y) \leqslant d(x, y) \leqslant c_2 d'(x, y)$ for all $x, y \in S$), so that every ball is open. In the following we shall assume that the quasimetric d is continuous.

As usual, we say w is a weight if w is a nonnegative locally integrable function. For a measurable set E, we denote $w(E) = \int_E w(y)\, d\mu(y)$. Thus, $w(E) = |E|_{w d\mu}$.

We also recall that the weight w is an $A_\infty(\mu)$ weight if there are postive constants C and δ such that

$$w(E) \leqslant C \left(\frac{\mu(E)}{\mu(B)}\right)^{\delta} w(B)$$

for every ball B and every measurable set $E \subset B$.

We denote by $d = \log C$ the doubling order of μ, where C is the smallest constant in (29). By iterating (29), we then have

$$\frac{r(\tilde{B})^d}{\mu(\tilde{B})} \leqslant C \frac{r(B)^d}{\mu(B)} \tag{30}$$

for every pair $\tilde{B}$, B of balls such that $\tilde{B} \subset B$.

We shall use a grid of dyadic sets in S which are "almost" balls as described [SW]. In fact, the following has been proved in [SW]:

If $\rho = 8K^5$, then for any (large negative) integer m, there are points $\{x_j^k\}$ and a family $\mathscr{D}_m = \{E_j^k\}$ of sets for $k = m, m+1, \ldots$ and $j = 1, 2, \ldots$ such that

- $B(x_j^k, \rho^k) \subset E_j^k \subset B(x_j^k, \rho^{k+1})$
- For each $k = m, m-1, ...,$ the family $\{E_j^k\}$ is pairwise disjoint in j,

and

$$S = \bigcup_j E_j^k.$$

- If $m \leqslant k < l$, then either $E_j^k \cap E_i^l = \varnothing$ or $E_j^k \subset E_i^l$.

We shall say that the family $\mathscr{D} = \bigcup_{m \in \mathbf{Z}} \mathscr{D}_m$ is a dyadic cube decomposition of S, and we call the sets in $\mathscr{D}$ dyadic cubes. A dyadic cube will usually be denoted by Q, and Q^* will denote the containing ball described above with $(1/\rho)\, Q^* \subset Q \subset Q^*$. We note that while the cubes in each $\mathscr{D}_m$ have the dyadic properties listed above, there are no nestedness properties of the cubes in $\mathscr{D}_{m_1}$ relative to the cubes in $\mathscr{D}_{m_2}$ if m_1, m_2 are different.

Following [W], we define

$$M_m g(x) = \sup_{\substack{B : x \in B \\ r(B) \geqslant \rho^m}} \frac{1}{\mu(B)} \int_B |g| \, d\mu,$$

and also the dyadic version

$$M_m^d g(x) = \sup_{\substack{Q : x \in Q \\ Q \in \mathscr{D}_m}} \frac{1}{\mu(Q)} \int_Q |g| \, d\mu.$$

We will use the following corollary of the Lebesgue Differentiation Theorem: if g is a nonnegative function in $L^1_{loc}(d\mu)$, then

$$g(x) \leqslant \liminf_{m \to -\infty} M_m^d g(x), \tag{31}$$

for a.e. $(d\mu)$ $x \in S$. Indeed, fix an x and for each m, let Q_m be the smallest cube in $\mathscr{D}_m$ which contains x. Since the cubes Q_m shrink to x, we can assume that

$$g(x) = \lim_{m \to -\infty} \frac{1}{\mu(Q_m)} \int_{Q_m} g \, d\mu,$$

this being true for a.e. $(d\mu)x$ by the Lebesgue Differentiation Theorem. Since $Q_m \in \mathscr{D}_m$, the last limit is clearly at most $\liminf_{m \to -\infty} M_m^d g(x)$, and (31) follows.

5. PROOFS

Before giving the proof of Theorem 2.3 we prove Corollary 2.4. Indeed, pick s with $p<s<r$. Using (12) with r replaced by s, we have

$$\frac{1}{w(B_0)}\int_{B_0} |f-f_{B_0}|^p \, w\, d\mu$$

$$=\frac{1}{w(B_0)}\int_0^\infty p\lambda^p w(\{x\in B_0 : |f(x)-f_{B_0}|>\lambda\})\,\frac{d\lambda}{\lambda}$$

$$\leqslant \frac{1}{w(B_0)}\int_0^\infty p\lambda^p \min\left\{w(B_0), \left(\frac{c\,\|a\|\,\|f\|_a\, a(\eta B_0)}{\lambda}\right)^s w(B_0)\right\}\frac{d\lambda}{\lambda}$$

$$\leqslant (c\,\|a\|\,\|f\|_a\, a(\eta B_0))^p + p\int_{c\,\|a\|\,\|f\|_a\, a(\eta B_0)}^\infty \lambda^{p-s}\{c\,\|a\|\,\|f\|_a\, a(\eta B_0)\}^s\,\frac{d\lambda}{\lambda}$$

$$\approx \|a\|^p\,\|f\|_a^p\, a(\eta B_0)^p. \quad \blacksquare$$

Proof of Theorems 2.3 *and* 2.7. It is enough to prove Theorem 2.7 since this includes Theorem 2.3. The only real difference in the two proofs occurs in Lemma 5.2 below. Observe that we may assume $\|f\|_a=1$ by renormalizing f if necessary. We must show that

$$w(\{x\in B_0 : |f(x)-f_{B_0}|>\lambda\})\leqslant \frac{c\,\|a\|^r}{\lambda^r}\, a'(\eta B_0)^r\, w(B_0) \tag{32}$$

with c independent of λ, f and B_0.

Letting $\tilde{f}=(f-f_{B_0})\,\chi_{B_0}$, we reduce everything to an estimate of $M_m^d(\tilde{f})$ in the appropriate norm, with norm constant independent of m for large negative m, since by (31) with $g=\tilde{f}$ and Fatou's Lemma,

$$\|\tilde{f}\|_{L^{r,\infty}(B_0, w)} \leqslant \liminf_{m\to-\infty}\, \|M_m^d(\tilde{f})\|_{L^{r,\infty}(B_0, w)}. \tag{33}$$

To estimate $\|M_m^d(\tilde{f})\|_{L^{r,\infty}(B_0, w)}$, we first show that, in a sense, it is possible to replace χ_{B_0} in the definition of $\tilde{f}$ by χ_{Q_0} for an appropriate dyadic cube Q_0 in $\mathscr{D}_m$ of approximately the same size as B_0. Let k_0 be the integer such that $\rho^{k_0}\leqslant r(B_0)<\rho^{k_0+1}$, and consider any m with $m<k_0$. We claim the following:

There are at most M cubes $E_j^{k_0}$ in $\mathscr{D}_m$ meeting B_0, where M is a structural constant which is independent of B_0 and m, provided that $m<k_0$.

Indeed, fix $m<k_0$ and denote the cubes in $\mathscr{D}_m$ which have nonempty intersection with B_0 by Q_j, $j=1, ..., M$. The Q_j are disjoint, and they are

contained in some fixed enlargement of B_0 since they touch B_0 and their radii are comparable to $r(B_0)$ (cf. (36) below). Thus, the sum of the $\mu(Q_j)$ is at most $c\mu(B_0)$. Since the size of each Q_j is comparable to the size of B_0, each $\mu(Q_j)$ exceeds a fixed multiple of $\mu(B_0)$ by doubling (in fact, the measures are comparable). Therefore, the number M of Q_j's must be at most a fixed geometric constant, which proves our claim.

Consequently, for any fixed $m < k_0$, if Q_j, $j = 1, ..., M$, are the cubes in $\mathscr{D}_m$ mentioned above, we obtain $\chi_{B_0} \leqslant \sum_{j=1}^{M} \chi_{Q_j}$, and so

$$M_m^d(\tilde{f}) \leqslant \sum_{j=1}^{M} M_m^d((f - f_{B_0})\,\chi_{Q_j})$$

and

$$\|M_m^d(\tilde{f})\|_{L^{r,\infty}(B_0,\,w)} \leqslant M \max_{j=1,\,...,\,M} \|M_m^d((f - f_{B_0})\,\chi_{Q_j})\|_{L^{r,\infty}(B_0,\,w)}. \tag{34}$$

For the rest of the proof, fix one of these cubes Q_j and denote it by Q_0. Let $\bar{f} = (f - f_{B_0})\,\chi_{Q_0}$. Then $\|M_m^d(\bar{f})\|_{L^{r,\infty}(B_0,\,w)}$ is a typical term in the max on the right side of (34), and we will now estimate this norm. By the first property of the dyadic cubes, there is a ball Q_0^* such that

$$\frac{1}{\rho} Q_0^* \subset Q_0 \subset Q_0^*$$

and $r(Q_0^*) = \rho^{k_0+1}$. We claim that if $\eta = 17K^7$, then

$$Q_0 \subset Q_0^* \subset \eta B_0. \tag{35}$$

Indeed, since $\varnothing \neq B_0 \cap Q_0 \subset B_0 \cap Q_0^*$ and $r(Q_0^*) \leqslant \rho r(B_0)$, (35) will follow from showing that if P and B are balls with $P \cap B \neq \varnothing$ and $r(P) \leqslant \beta r(B)$ for some $\beta > 0$, then

$$P \subset c_\beta B \tag{36}$$

with $c_\beta = K^2 + K\beta + K^2\beta$. (Note that $c_\beta \leqslant 17K^7$ when $\beta = \rho$, by definition of ρ.) To verify (36), note that if $z \in B \cap P$ and $y \in P$, then

$$\begin{aligned} d(y, x_B) &\leqslant K(d(y, x_P) + d(x_P, x_B)) \leqslant K(r(P) + K(d(x_P, z) + d(z, x_B))) \\ &\leqslant K(r(P) + K(r(P) + r(B))) \leqslant K(\beta r(B) + K(\beta r(B) + r(B))) \\ &= K^2\left(1 + \frac{\beta}{K} + \beta\right) r(B), \end{aligned}$$

which implies (36).

Recall that $\bar{f} = (f - f_{B_0})\chi_{Q_0}$. For each $\lambda > 0$, let Ω_λ be the set

$$\Omega_\lambda = \{x \in S : M_m^d(\bar{f})(x) > \lambda\}.$$

The following observation will be used. If g is a nonnegative function supported in Q_0, then for each λ which satisfies

$$\lambda \geqslant \frac{1}{\mu(Q_0)} \int_{Q_0} g \, d\mu,$$

we have

$$\{x \in S : M_m^d g(x) > \lambda\} = \{x \in Q_0 : M_m^d g(x) > \lambda\} \subset Q_0. \tag{37}$$

In fact, if $x \notin Q_0$ then by the dyadic structure, since $Q_0 \in \mathscr{D}_m$,

$$M_m^d g(x) = \max\left\{ \sup_{\substack{Q : x \in Q \in \mathscr{D}_m \\ Q \subset Q_0}} \frac{1}{\mu(Q)} \int_Q g \, d\mu, \sup_{\substack{Q : x \in Q \in \mathscr{D}_m \\ Q \supset Q_0}} \frac{1}{\mu(Q)} \int_Q g \, d\mu \right\}$$

$$= \sup_{\substack{Q : x \in Q \in \mathscr{D}_m \\ Q \supset Q_0}} \frac{1}{\mu(Q)} \int_Q g \, d\mu$$

since $x \notin Q_0$

$$= \sup_{\substack{Q : x \in Q \in \mathscr{D}_m \\ Q \supset Q^0}} \frac{1}{\mu(Q)} \int_{Q_0} g \, d\mu = \frac{1}{\mu(Q_0)} \int_{Q_0} g \, d\mu \leqslant \lambda$$

since the support of g lies in Q_0. This yields (37).

For fixed $\lambda > 0$, let $\{Q_i\}$ be the maximal (with respect to inclusion) dyadic cubes in $\mathscr{D}_m$ satisfying

$$\lambda < \frac{1}{\mu(Q_i)} \int_{Q_i} |\bar{f}| \, d\mu.$$

It follows easily from the maximality of the cubes and from the doubling of μ that

$$\lambda < \frac{1}{\mu(Q_i)} \int_{Q_i} |\bar{f}| \, d\mu \leqslant C\lambda \tag{38}$$

for each i. Also, because they are maximal dyadic cubes, the Q_i are pairwise disjoint, and we have

$$\Omega_\lambda = \bigcup_i Q_i. \tag{39}$$

Moreover, by the observation above, with g taken to be $|\bar{f}| = |f - f_{B_0}| \chi_{Q_0}$,

$$Q_i \subset Q_0 \; (\Omega_\lambda \subset Q_0) \tag{40}$$

if λ satisfies

$$\lambda \geqslant \frac{1}{\mu(Q_0)} \int_{Q_0} |f - f_{B_0}| \, d\mu.$$

The next lemma contains a key inequality of good-λ type relating M_m^d and the following local maximal operator associated with the functional a: given a ball B_0, define

$$A_{B_0}(x) = \sup_{B:\, x \in B \subset B_0} a(B),$$

where the supremum is taken over all balls B contained in B_0 and containing x.

In stating the lemma, we use the same notation as above.

LEMMA 5.1. *Let $w \in A_\infty(\mu)$ and let f satisfy* (11) *with $\|f\|_a = 1$. There exist geometric constants N, $C > 1$ and $\varepsilon_0 > 0$ such that for all λ, ε with $\lambda > 0$ and $0 < \varepsilon < \varepsilon_0$,*

$$w(\eta B_0 \cap \Omega_{N\lambda}) \leqslant C\varepsilon^\delta w(\eta B_0 \cap \Omega_\lambda) + w(\{x \in \eta B_0 : A_{\eta B_0}(x) > \varepsilon\lambda\}), \tag{41}$$

where δ depends on the A_∞ constant of w. All constants are independent of m, f, Q_0, B_0, and λ.

The basic estimate which follows in a standard way from this good-λ inequality is

$$\|M_m^d(\bar{f})\|_{L^{r,\infty}(\eta B_0, w)} \leqslant C \, \|A_{\eta B_0}\|_{L^{r,\infty}(\eta B_0, w)},$$

with constant C independent of m. Recall that $\bar{f} = (f - f_{B_0}) \chi_{Q_0}$ and Q_0 is one of the M dyadic cubes $E_j^{k_0}$ which cover B_0. We then obtain from (34) that

$$\|M_m^d(\tilde{f})\|_{L^{r,\infty}(B_0, w)} \leqslant MC \, \|A_{\eta B_0}\|_{L^{r,\infty}(\eta B_0, w)}, \tag{42}$$

with constants independent of m if $m < k_0$.

The control of $A_{\eta B_0}$ can be deduced from the following lemma.

LEMMA 5.2. *Let B be a ball and let a be a functional for which* (14) *holds for some doubling weight w. Then there exists a positive constant C, independent of B, such that for all $\lambda > 0$,*

$$w(\{x \in B : A_B(x) > \lambda\}) \leqslant \frac{C \, \|a\|^r}{\lambda^r} a'(B)^r \, w(B),$$

or equivalently,

$$\|A_B\|_{L^{r,\infty}(B,w)} \leqslant C \, \|a\| \, a'(B).$$

Theorem 2.7 (and so also Theorem 2.3) follows by combining (33) and (42) with Lemma 5.2,

$$\begin{aligned}\|\tilde{f}\|_{L^{r,\infty}(B_0,w)} &\leqslant \liminf_{m \to -\infty} \|M_m^d(\tilde{f})\|_{L^{r,\infty}(B_0,w)} \\ &\leqslant MC \, \|A_{\eta B_0}\|_{L^{r,\infty}(\eta B_0,w)} \leqslant C \, \|a\| \, a'(\eta B_0)\end{aligned}$$

as desired. (Recall that we are assuming that $\|f\|_a = 1$.)

We now prove the good-λ inequality.

Proof of Lemma 5.1. First observe that the conclusion holds whenever

$$\lambda \leqslant \frac{1}{\mu(Q_0)} \int_{Q_0} |f - f_{B_0}| \, d\mu.$$

Indeed, for such λ, since μ is doubling and $(1/\rho) \, Q_0^* \subset Q_0 \subset Q_0^* \subset \eta B_0$ by (35), and since Q_0^* and ηB_0 have radii of comparable sizes,

$$\begin{aligned}\lambda &\leqslant \frac{C}{\mu(\eta B_0)} \int_{\eta B_0} |f - f_{B_0}| \, d\mu \leqslant \frac{C}{\mu(\eta B_0)} \int_{\eta B_0} |f - f_{\eta B_0}| \, d\mu + C \, |f_{\eta B_0} - f_{B_0}| \\ &\leqslant \frac{C}{\mu(\eta B_0)} \int_{\eta B_0} |f - f_{\eta B_0}| \, d\mu \leqslant Ca(\eta B_0) \leqslant C \inf_{x \in \eta B_0} A_{\eta B_0}(x).\end{aligned}$$

The next-to-last inequality above follows from (11). Thus we can write

$$\begin{aligned}w(\eta B_0 \cap \Omega_{N\lambda}) &\leqslant w(\eta B_0) = w(\{x \in \eta B_0 : A_{\eta B_0}(x) > \lambda/C\}) \\ &\leqslant w(\{x \in \eta B_0 : A_{\eta B_0}(x) > \varepsilon\lambda\}) + C\varepsilon^\delta w(\eta B_0 \cap \Omega_\lambda)\end{aligned}$$

if $\varepsilon < 1/C$. Therefore we may assume from now on that

$$\lambda > \frac{1}{\mu(Q_0)} \int_{Q_0} |f - f_{B_0}| \, d\mu,$$

and so (40) holds.

Now since $N > 1$, $\Omega_{N\lambda} \subset \Omega_\lambda$. Thus, by (39) and the fact that the cubes $\{Q_i\}$ are pairwise disjoint, we have

$$\begin{aligned} w(\eta B_0 \cap \Omega_{N\lambda}) &= w(\eta B_0 \cap \Omega_{N\lambda} \cap \Omega_\lambda) \\ &= \sum_i w(\{x \in \eta B_0 \cap Q_i : M_m^d(\bar{f})(x) > N\lambda\}) \\ &\leqslant \sum_i w(\{x \in Q_i : M_m^d(\bar{f})(x) > N\lambda\}). \end{aligned}$$

For each fixed i, we claim that

$$w(\{x \in Q_i : M_m^d(\bar{f})(x) > N\lambda\}) = w(\{x \in Q_i : M_m^d(\bar{f}\chi_{Q_i})(x) > N\lambda\}).$$

Indeed, let $x \in Q_i \cap P$ where P is a dyadic cube in $\mathscr{D}_m$ containing x. Then by the nestedness property of the dyadic cubes, there are two possibilities:

(i) either $P \subset Q_i$, and therefore

$$\frac{1}{\mu(P)} \int_P |\bar{f}| \, d\mu = \frac{1}{\mu(P)} \int_P |\bar{f}| \, \chi_{Q_i} \, d\mu \leqslant M_m^d(\bar{f}\chi_{Q_i})(x),$$

or

(ii) $Q_i \subset P$, so that

$$\frac{1}{\mu(P)} \int_P |\bar{f}| \, d\mu \leqslant \lambda$$

by the maximality property of Q_i. If $M_m^d(\bar{f})(x) > N\lambda$, observe that the cubes P of the second type do not count in computing $M_m^d(\bar{f})(x)$ since $N > 1$. This proves our claim.

On the other hand, by the properties of the cubes we get

$$\begin{aligned} |f(x) - f_{B_0}| &\leqslant |f(x) - f_{Q_i}| + |f_{Q_i} - f_{B_0}| \\ &\leqslant |f(x) - f_{Q_i}| + \frac{1}{\mu(Q_i)} \int_{Q_i} |f - f_{B_0}| \, d\mu \\ &= |f(x) - f_{Q_i}| + \frac{1}{\mu(Q_i)} \int_{Q_i} |\bar{f}| \, d\mu \end{aligned}$$

since $Q_i \subset Q_0$ by (40)

$$\leqslant |f(x) - f_{Q_i}| + C\lambda$$

by (38). Then defining

$$E_i = \{x \in Q_i : M_m^d((f - f_{Q_i})\, \chi_{Q_i})(x) > (N - C)\, \lambda\},$$

if $N > C$, we have

$$\{x \in Q_i : M_m^d(\bar{f}\chi_{Q_i})(x) > N\lambda\} \subset E_i,$$

and consequently

$$w(\eta B_0 \cap \Omega_{N\lambda}) \leqslant \sum_{i:\ Q_i \subset Q_0} w(E_i).$$

For each of these dyadic cubes Q_i, recall that Q_i^* is the ball so that

$$\frac{1}{\rho}\, Q_i^* \subset Q_i \subset Q_i^*,$$

where $\rho = 8K^5$, and therefore, by the doubling property of w, $w(Q_i)$ and $w(Q_i^*)$ are comparable uniformly in i.

For each ε with $0 < \varepsilon < 1/C$, we split the index set in two: we say that $i \in I$ if

$$a(Q_i^*) \leqslant \varepsilon\lambda$$

and $i \in II$ if

$$a(Q_i^*) > \varepsilon\lambda.$$

Then

$$w(\eta B_0 \cap \Omega_{N\lambda}) \leqslant \sum_{i \in I} w(E_i) + \sum_{i \in II} w(E_i) = I + II.$$

For I, we use the fact that M_m^d is of weak type (1, 1) with weak type constant 1, together with the basic hypothesis (11) (which we observe is stated with respect to balls), to first control the unweighted measure of E_i:

$$\mu(E_i) \leqslant \frac{1}{\lambda}\, \frac{1}{\mu(Q_i)} \int_{Q_i} |f - f_{Q_i}|\, d\mu\, \mu(Q_i)$$

$$\leqslant \frac{C}{\lambda}\, \frac{1}{\mu(Q_i^*)} \int_{Q_i^*} |f - f_{Q_i^*}|\, d\mu\, \mu(Q_i^*) \leqslant \frac{C}{\lambda}\, a(Q_i^*)\, \mu(Q_i^*) \leqslant C\varepsilon\mu(Q_i^*).$$

Then because $w \in A_\infty(\mu)$, we obtain for each $i \in I$ that

$$w(E_i) \leqslant C\varepsilon^\delta w(Q_i^*)$$

for some $C, \delta > 0$ which are independent of i. Therefore,

$$I \leqslant C\varepsilon^\delta \sum_{i \in I} w(Q_i^*) \leqslant C\varepsilon^\delta \sum_{i \in I} w(Q_i) \leqslant C\varepsilon^\delta w(Q_0 \cap \Omega_\lambda)$$

since w is doubling and the cubes $\{Q_i\}$ are pairwise disjoint and contained in Q_0 (by (40) and the restriction on λ). Hence,

$$I \leqslant C\varepsilon^\delta w(\eta B_0 \cap \Omega_\lambda)$$

since $Q_0 \subset \eta B_0$.

To estimate II, we recall again that $Q_i \subset Q_i^* \subset \eta B_0$. Then if $x \in Q_i$, $A_{\eta B_0}(x) > a(Q_i^*) > \varepsilon\lambda$, so that

$$\begin{aligned} II &\leqslant \sum_{i \in II} w(Q_i) = \sum_{i \in II} w(\{x \in Q_i : A_{\eta B_0}(x) \geqslant \varepsilon\lambda\}) \\ &\leqslant Cw(\{x \in Q_0 : A_{\eta B_0}(x) \geqslant \varepsilon\lambda\}) \\ &\leqslant Cw(\{x \in \eta B_0 : A_{\eta B_0}(x) > \varepsilon\lambda\}) \end{aligned}$$

since $Q_0 \subset \eta B_0$.

Combining these estimates, we have for $\lambda > 0$ and small $\varepsilon > 0$ that

$$w(\eta B_0 \cap \Omega_{N\lambda}) \leqslant C\varepsilon^\delta w(\eta B_0 \cap \Omega_\lambda) + Cw(\{w \in \eta B_0 : A_{\eta B_0}(x) > \varepsilon\lambda\}), \quad (43)$$

where C is a structural constant. This proves the desired good-λ inequality (41) and completes the proof of Lemma 5.1.

We are left with the proof of the second lemma.

Proof of Lemma 5.2. The proof is by a standard covering lemma. If x is in the set $\{x \in B: A_B(x) > \lambda\}$, then for some ball $P \subset B$ with $x \in P$, we have $\lambda < a(P)$. Pick a Vitali type cover of $\{x \in B: A_B(x) > \lambda\}$ by such balls $\{P_i\}$. Then the balls $\{P_i\}$ are pairwise disjoint subballs of B and $\{x \in B: A_B(x) > \lambda\} \subset \bigcup_i cP_i$. Therefore,

$$\begin{aligned} w(\{x \in B: A_B(x) > \lambda\}) &\leqslant \sum_i w(cP_i) \leqslant C \sum_i w(P_i) \leqslant \frac{C}{\lambda^r} \sum_i a(P_i)^r\, w(P_i) \\ &\leqslant \frac{C\,\|a\|^r}{\lambda^r}\, a'(B)^r\, w(B) \end{aligned}$$

by (14) since $\{P_i\}$ is a pairwise disjoint family of subballs of B.

This completes the proof of Lemma 5.2, and hence the proof of Theorem 2.7 is also complete. ∎

Proof of Theorem 3.1. The point of departure of this theorem is the following information on $\mathscr{F}$ and $\mathscr{B}$,

$$\frac{1}{\mu(B)}\int_B |f-f_B|\,d\mu \leqslant cb(B,f), \tag{44}$$

with c independent of $f\in\mathscr{F}$ and $B\subset\eta B_0$. We consider only functions f in $\mathscr{F}$.

Then, by Theorem 2.3 and the assumption (22) on b, we have the weak type estimate

$$\sup_{t>0} t^r\,\frac{w(\{x\in B_0 : |f-f_{B_0}|>t\})}{w(B_0)} \leqslant C^r b(\eta B_0, f)^r. \tag{45}$$

What we need to do is pass from this weak type inequality to the corresonding strong type estimate. To accomplish this, we will adapt some ideas of R. Long and F. Nie in [LN]. Since $b(B_0,f)=b(B_0,f+\lambda)$ for any λ (due to the conditions on b), we may assume that $f_{B_0}=0$. To simplify notation, we let $g=|f|$, so that $g\in\mathscr{F}$ by our assumptions about $\mathscr{F}$.

Let $\lambda>0$ be a positive number to be chosen and set $\lambda_k=\lambda 2^k$ for $k=0, 1, 2, \ldots$. Note that $\lambda_{k+1}=2\lambda_k$. Then

$$\begin{aligned}
&\frac{1}{w(B_0)}\int_{B_0} g^r w\,d\mu\\
&\quad=\frac{1}{w(B_0)}\int_{\{B_0\,:\,g\leqslant 4\lambda\}} g^r w\,d\mu+\frac{1}{w(B_0)}\sum_{k=1}^{\infty}\int_{\{B_0\,:\,\lambda_{k+1}<g\leqslant\lambda_{k+2}\}} g^r w\,d\mu\\
&\quad\leqslant 4^r\lambda^r+\frac{4^r}{w(B_0)}\sum_{k=1}^{\infty}\lambda_{k-1}^r w(F_{k+1}),
\end{aligned}$$

where for each integer k, we define $F_k=\{x\in B_0 : \lambda_k<g(x)\leqslant\lambda_{k+1}\}$.

Now, if we recall the notation

$$\tau_\lambda(g)=\min\{g, 2\lambda\}-\min\{g,\lambda\},$$

we have for $x\in F_{k+1}$, $k=1, 2, \ldots,$

$$\begin{aligned}
\lambda_k=\tau_{\lambda_k}(g)(x)&\leqslant|\tau_{\lambda_k}(g)(x)-(\tau_{\lambda_k}(g))_{B_0}|+(\tau_{\lambda_k}(g))_{B_0}\\
&\leqslant|\tau_{\lambda_k}(g)(x)-(\tau_{\lambda_k}(g))_{B_0}|+g_{B_0}
\end{aligned}$$

since $\tau_\lambda(g) \leqslant g$. If we choose $\lambda = 2g_{B_0}$, the last inequality yields

$$\lambda_k \leqslant |\tau_{\lambda_k}(g)(x) - (\tau_{\lambda_k}(g))_{B_0}| + \frac{\lambda}{2} < |\tau_{\lambda_k}(g)(x) - (\tau_{\lambda_k}(g))_{B_0}| + \frac{\lambda_k}{2},$$

and therefore, if $x \in F_{k+1}$,

$$\lambda_{k-1} < |\tau_{\lambda_k}(g)(x) - (\tau_{\lambda_k}(g))_{B_0}|.$$

Finally, by combining inequalities and using the weak type estimate (45) for each $\tau_{\lambda_k}(g)$, we get

$$\begin{aligned}
&\frac{1}{w(B_0)} \int_{B_0} g^r w \, d\mu \\
&\quad \leqslant C\lambda^r + \frac{C}{w(B_0)} \sum_{k=1}^{\infty} \lambda_{k-1}^r w(\{x \in B_0 : |\tau_{\lambda_k}(g)(x) - (\tau_{\lambda_k}(g))_{B_0}| > \lambda_{k-1}\}) \\
&\quad \leqslant C\lambda^r + C^r \sum_{k=1}^{\infty} b(\eta B_0, \tau_{\lambda_k}(g))^r \\
&\quad \leqslant C\lambda^r + C^r b(\eta B_0, g)^r \leqslant C^r b(\eta B_0, |f|)^r \leqslant C^r b(\eta B_0, f)^r,
\end{aligned}$$

by (21) and (44). This concludes the proof of the theorem. ∎

Only minor changes in the proof above are needed in order to prove the more general version of Theorem 3.1 mentioned in Remark 3.3. For example, in the more general situation, instead of (45) we would have the analogous inequality with b replaced by b' on the right side. The remaining changes are simple and we shall not give the details.

Proof of Corollary 3.2. We first observe that the functional on the right side in (25) is majorized by

$$b(B, f) = r(B) \left(\frac{1}{|B|_{v\,dv}} \int_B |Xf|^p \, v \, dv \right)^{1/p};$$

in fact, this follows by using Hölder's inequality together with the fact that $v \in A_{p/p_0}(v)$. We are left with checking that b satisfies the appropriate versions of the two conditions (21) and (22), namely those with $r = q$ and $w(B) = |B|_{w\,d\mu}$ for any ball B. The former follows easily as in the main example given after (21). For the latter, we have to prove that

$$\sum_i b(B_i, f)^q \, |B_i|_{w\,d\mu} \leqslant Cb(B, f)^q \, |B|_{w\,d\mu} \tag{46}$$

for every f and B whenever $\{B_i\}$ is a pairwise disjoint family of subballs of B. Indeed,

$$\sum_i b(B_i, f)^q \, |B_i|_{w\,d\mu}$$

$$= |B|_{w\,d\mu}\, r(B)^q \sum_i \frac{r(B_i)^q}{r(B)^q} \left(\frac{1}{|B_i|_{v\,dv}} \int_{B_i} |Xf|^p \, v\, dv \right)^{q/p} \frac{|B_i|_{w\,d\mu}}{|B|_{w\,d\mu}}$$

$$\leqslant C\, |B|_{w\,d\mu}\, r(B)^q \sum_i \left(\frac{|B_i|_{v\,dv}}{|B|_{v\,dv}} \right)^{q/p} \left(\frac{1}{|B_i|_{v\,dv}} \int_{B_i} |Xf|^p \, v\, dv \right)^{q/p}$$

$$\leqslant C\, \frac{|B|_{w\,d\mu}\, r(B)^q}{|B|_{v\,dv}^{q/p}} \left(\sum_i \int_{B_i} |Xf|^p \, v\, dv \right)^{q/p} \leqslant C b(B, f)^q \, |B|_{w\,d\mu},$$

as desired, where we have used the balance condition (26) to obtain the first inequality, and the disjointness of the B_i's to obtain the last. ∎

We close this section by making some further comments about Remark 2.6 concerning why it is possible to take $\eta = 1$ in the conclusion of Theorem 2.3 in case B_0 satisfies the Boman chain condition and $a(B)$ satisfies (9) for all collections $\{B_i\}$ of subballs of B with bounded overlaps, i.e., for all such collections with

$$\sum_i \chi_{B_i}(x) \leqslant N \chi_B(x)$$

for all x, where N is an appropriately large fixed positive number. In fact, this phenomenon is not limited to balls B_0 but holds for any Boman domain Ω of type $F(\sigma, N)$: see, e.g., [C] or [FGuW] for the precise definition of such domains. The main result that we need is the following analogue of Theorem 1.5 of [C]. We use the notation

$$\|g\|_{\mathscr{L}^{r,\infty}_w(\Omega)} = \sup_{\lambda > 0} \lambda w(\{x \in \Omega \colon |g| > \lambda\})^{1/r}$$

for the (unnormalized) weak $L^r_w(\Omega)$ norm, $1 < r < \infty$.

LEMMA 5.3. *Let (S, d, μ) be a space of homogeneous type. Let $\sigma, N \geqslant 1$, $1 < r < \infty$, $\Omega \in F(\sigma, N)$ and f be a measurable function on Ω. Let w be a doubling measure and suppose that for each ball B with $\sigma B \subset \Omega$, there is a constant f_B such that*

$$\|f - f_B\|_{\mathscr{L}^{r,\infty}_w(B)} \leqslant \alpha(\sigma B),$$

where $\alpha(B)$ is a nonnegative functional. Then there exists a constant f_Ω such that

$$\|f - f_\Omega\|_{\mathscr{L}_w^{r,\infty}(\Omega)} \leqslant c \sup \left(\sum \alpha(\sigma B)^r \right)^{1/r},$$

where the sup *is taken over all countable collections $\{B\}$ of balls such that $\{\sigma B\}$ has overlaps bounded by N and $\sigma B \subset \Omega$, and c is a constant depending only on r, w, σ, N, and μ.*

The main differences between Lemma (5.3) and the situation in [C] are the facts that weak norms rather than strong norms appear above, and the functional $\alpha(B)$ is not stipulated. The proof of the lemma, however, is virtually identical to the proof of Theorem 1.5 in [C], and we shall only make a few remarks about the differences. First, we note that $\|\cdot\|_{\mathscr{L}_w^{r,\infty}}$ is a norm since $1 < r < \infty$. Moreover, we need the following analogue for weak norms of Lemma 2.5 of [C]: if $1 < p < \infty$, $\{B_i\}$ is any collection of balls and $\{c_i\}$ is any sequence of nonnegative numbers, then

$$\left\|\sum c_i \chi_{NB_i}\right\|_{\mathscr{L}_w^{p,\infty}(S)} \leqslant C(w, p, N, \mu) \left\|\sum c_i \chi_{B_i}\right\|_{\mathscr{L}_w^{p,\infty}(S)}$$

for any doubling measure μ.

The proof of this estimate follows by a simple duality argument as in [C], using the fact that the homogeneous Hardy–Littlewood maximal function M_w is a bounded operator on the Lorentz space $\mathscr{L}_w^{p',1}(S)$ dual to $\mathscr{L}_w^{p,\infty}(S)$, $1/p + 1/p' = 1$, $1 < p < \infty$: see, e.g., [BS], Theorem 4.13 and Remark 4.15.

Finally, to obtain the conclusion of Theorem 2.3 with $\eta = 1$, namely

$$\|f - f_\Omega\|_{\mathscr{L}_w^{r,\infty}(\Omega)} \leqslant ca(\Omega)\, w(\Omega)^{1/r},$$

we use the functional α defined by $\alpha(B) = w(B)^{1/r}\, a(B)$. The fact that w is doubling means that $\alpha(\sigma B)^r \leqslant Cw(B)\, a(\sigma B)^r$, and the desired conclusion then follows from Theorem 2.3, Lemma 5.3 and our assumption that (9) holds for all collections $\{B_i\}$ of balls with bounded overlaps.

6. EXAMPLES

6.1. *D_r Does Not Imply $D_{r+\varepsilon}$ in General*

In this section we shall consider $\mathbf{R}$ with the usual distance and endowed with Lebesgue measure, and we consider functionals

$$a\colon \mathscr{I} \to (0, \infty),$$

where $\mathscr{I}$ is the set of all finite intervals. Let $r>1$ and recall that a satisfies the (unweighted) D_r condition if there exists a finite constant C such that for each interval I and for every family $\Delta=\Delta(I)$ of disjoint subintevals P of I we have

$$\sum_{P\in\Delta} a(P)^r\,\ell(P)\leqslant C^r a(I)^r\,\ell(I),$$

where $\ell(P)$ denotes the length of P.

Consider the functional

$$a(I)=\frac{\ell(I)^\alpha}{\ell(I)}\int_I g(x)\,dx,$$

where $0\leqslant g\in L^1_{loc}(\mathbf{R})$ and $0<\alpha<1$. Let $r=1/(1-\alpha)>1$. Then the easy but key computation done in (10) shows that $a\in D_r$, no matter which g we take. However, we cannot expect any better in general due to the following fact:

There exists a locally integrable function g such that for all positive ε,

$$a\notin D_{r+\varepsilon}.$$

The function g cannot belong to $A_\infty(\mathbf{R})$ by Corollary 2.10.

To verify this fact, let us consider the unit interval $I=(0,1)$ and the disjoint dyadic subintervals $I_k=(1-(1/2^{k-1}),1-(1/2^k))$, $k=1,2,\ldots,$ and define the function

$$g(x)=\begin{cases} a_k & \text{if}\quad x\in I_k\\ 0 & \text{otherwise,}\end{cases}$$

where the sequence $\{a_k\}$ is chosen so that $\{a_k/2^k\}\in\ell^1$. This condition implies that $g\in L^1(\mathbf{R})$, and therefore in particular that $a\in D_r$ and

$$\sum_{k=1}^{\infty} a(I_k)^r\,\ell(I_k)\leqslant C^r a(I)^r\,\ell(I).$$

On the other hand,

$$\sum_{k=1}^{\infty} a(I_k)^{r+\varepsilon}\,\ell(I_k)=\sum_{k=1}^{\infty}\left(\int_{I_k} g\right)^{r+\varepsilon} 2^{k\varepsilon/r}=\sum_{k=1}^{\infty}\left(\frac{a_k}{2^k}\right)^{r+\varepsilon} 2^{k\varepsilon/r}.$$

It is enough to take for instance $a_k=2^k/k^{1+\delta}$ with $\delta>0$ to see that the last series diverges for any $\varepsilon>0$. This shows that $a\notin D_{r+\varepsilon}$ for any $\varepsilon>0$.

Variations of this example can be given in $\mathbf{R}^n$.

6.2. *An Example Where D_r Implies $D_{r+\varepsilon}$*

We give the proof here of Corollary 2.10. Consider the fractional functional on a space (S, d, μ) of homogeneous type:

$$a(B) = r(B)^{\alpha} \left(\frac{1}{\mu(B)} \int_B g \, d\mu \right)^{1/p_0},$$

where $\alpha > 0$, $1 \leqslant p_0 < d/\alpha$ and d is the doubling order of μ. For any locally integrable $g \geqslant 0$, this functional belongs to unweighted D_r with $r = p_0 d/(d - \alpha p_0)$, as can be seen by using an argument like the one we gave in case $p_0 = 1$ in Section 2. We now show that if we make the stronger assumption that $g \in A_\infty(\mu)$, then $a \in D_{r+\varepsilon}$ for r as above and some $\varepsilon > 0$.

By standard properties of the theory of $A_\infty(\mu)$ weights, there exists a constant $t > 1$ such that

$$\left(\frac{1}{\mu(B)} \int_B g^t \, d\mu \right)^{1/t} \leqslant C \frac{1}{\mu(B)} \int_B g \, d\mu.$$

Let $s = tp_0 d/(d - \alpha t p_0)$ and note that $s > r$ since $t > 1$. Since $s > p_0$, we may assume that $s > p_0 t$ by choosing t sufficiently close to 1. We claim that

$$a \in D_s = D_{r+\varepsilon} \, (w = 1).$$

Indeed, if B_i and B are balls with $B_i \subset B$, we have by Hölder's inequality and doubling that

$$\begin{aligned} a(B_i)^s \, \mu(B_i) &\leqslant \left(\int_{B_i} g^t \, d\mu \right)^{s/p_0 t} \left(\frac{r(B_i)^d}{\mu(B_i)} \right)^{(s/p_0 t) - 1} \\ &\leqslant C \left(\frac{r(B)^d}{\mu(B)} \right)^{(s/p_0 t) - 1} \left(\int_{B_i} g^t \, d\mu \right)^{s/p_0 t}, \end{aligned}$$

and consequently, if $\{B_i\}$ is a pairwise disjoint collection of balls in B,

$$\begin{aligned} \sum_i a(B_i)^s \, \mu(B_i) &\leqslant C \left(\frac{r(B)^d}{\mu(B)} \right)^{(s/p_0 t) - 1} \sum_i \left(\int_{B_i} g^t \, d\mu \right)^{s/p_0 t} \\ &\leqslant C \left(\frac{r(B)^d}{\mu(B)} \right)^{(s/p_0 t) - 1} \left(\sum_i \int_{B_i} g^t \, d\mu \right)^{s/p_0 t} \\ &\leqslant C \left(\frac{r(B)^d}{\mu(B)} \right)^{(s/p_0 t) - 1} \left(\int_B g^t \, d\mu \right)^{s/p_0 t} \\ &\leqslant C \left(\frac{r(B)^d}{\mu(B)} \right)^{(s/p_0 t) - 1} \mu(B)^{s/p_0 t} \left(\frac{1}{\mu(B)} \int_B g \, d\mu \right)^{s/p_0} \\ &= C a(B)^s \, \mu(B). \end{aligned}$$

6.3. *The Weak Fractional Functional in the General Situation*

Let $\lambda > 1$ be fixed and consider the weak fractional functional on a space of homogeneous type (S, d, μ),

$$a(B) = r(B)^{\alpha} \left(\frac{v(\lambda B)}{\mu(B)} \right)^{1/p_0},$$

where $\alpha > 0$, $1 \leqslant p_0 < d/\alpha$ and d is the doubling order of μ. We claim that for $1 < r < p_0 d/(d - \alpha p_0)$,

$$\sum_i a(B_i)^r \, \mu(B_i) \leqslant c a'(B)^r \, \mu(B) \tag{47}$$

whenever $\{B_i\}$ is a collection of pairwise disjoint balls in B, were a' is defined by

$$a'(B) = r(B)^{\alpha} \left(\frac{v(\lambda' B)}{\mu(B)} \right)^{1/p_0} \approx a(\lambda' B)$$

for an appropriate value $\lambda' \geqslant \lambda$.

Setting $\varepsilon = (p_0 d/r) - (d - \alpha p_0)$, we have $\varepsilon > 0$ for such r. Since d is the doubling order of μ, we easily obtain (we may assume that $p_0 < r$ by nestedness of D_r) that for any B_i in a collection $\{B_i\}$ of pairwise disjoint balls in B,

$$a(B_i)^r \, \mu(B_i) \leqslant c \left(\frac{r(B)^d}{\mu(B)} \right)^{(r/p_0) - 1} \{ r(B_i)^{\varepsilon} \, v(\lambda B_i) \}^{r/p_0}.$$

Adding over i and using the fact that $r > p_0$, we have

$$\sum a(B_i)^r \, \mu(B_i) \leqslant c \left(\frac{r(B)^d}{\mu(B)} \right)^{(r/p_0) - 1} \left\{ \sum r(B_i)^{\varepsilon} \, v(\lambda B_i) \right\}^{r/p_0}.$$

Let x_i denote the center of B_i, and rewrite the last sum as

$$\sum r(B_i)^{\varepsilon} \, v(\lambda B_i) = \sum r(B_i)^{\varepsilon} \int_{d(x_i, y) < \lambda r(B_i)} dv(y)$$

$$= \int_{\bigcup (\lambda B_i)} h(y) \, dv(y),$$

where $h(y)$ is defined by

$$h(y) = \sum_{i:\, d(x_i,\, y) < \lambda r(B_i)} r(B_i)^{\varepsilon}.$$

We will show that $h(y) \leqslant cr(B)^{\varepsilon}$ for all y. We begin by noting that, since $r(B_i) \leqslant cr(B)$ for all i, we may divide the collection of indices i into sets

$$C_k = \left\{ i : 2^{-k-1} < \frac{r(B_i)}{cr(B)} \leqslant 2^k \right\}$$

for $k = 0, 1, \ldots$. Now, given any y, if i satisfies $d(x_i, y) < \lambda r(B_i)$, then $B_i \subset B(y, c\lambda r(B_i))$ for some constant c which is independent of y and i. Thus,

$$h(y) = \sum_k \sum_{\substack{i \in C_k \\ B_i \subset B(y,\, c\lambda 2^{-k} r(B))}} r(B_i)^{\varepsilon}.$$

Since the B_i are disjoint and since the radii of B_i and $B(y, c\lambda 2^{-k} r(B))$ are comparable when $i \in C_k$, with constants of comparability which are independent of i and k, it follows from the doubling of μ that the number of terms in the inner sum above is bounded by a constant N independent of k. Thus

$$h(y) \leqslant \sum_k N(c2^{-k} r(B))^{\varepsilon} \leqslant cr(B)^{\varepsilon},$$

as desired.

We then obtain by combining estimates that

$$\begin{aligned} \sum a(B_i)^r \mu(B_i) &\leqslant c \left(\frac{r(B)^d}{\mu(B)} \right)^{(r/p_0)-1} \{ r(B)^{\varepsilon} \nu(\cup (\lambda B_i)) \}^{r/p_0} \\ &= c \left\{ \frac{r(B)^{\alpha}}{\mu(B)^{1/p_0}} \nu(\cup (\lambda B_i))^{1/p_0} \right\}^r \mu(B). \end{aligned}$$

Finally, since $\bigcup (\lambda B_i) \subset \lambda' B$ for an appropriate value $\lambda' \geqslant \lambda$, we obtain the desired condition. Note that in Euclidean space with the usual Euclidean metric we can pick $\lambda' = \lambda$.

APPENDIX: THE EUCLIDEAN CASE AND POLYNOMIAL APPROXIMATION

In this appendix, we shall consider the space of homogeneous type $(\mathbf{R}^n, \mu, d)$, where d is Euclidean distance and μ is a doubling measure with doubling constant C_μ. In this case, our main results and their proofs are somewhat simpler. We denote by $\mathcal{Q}$ the family of all cubes in $\mathbf{R}^n$ with sides parallel to the axes, and we consider functionals $a\colon \mathcal{Q} \to (0, \infty)$.

We first give a dyadic version of Definition 2.1.

DEFINITION 7.1. Let $1 \leqslant r < \infty$ and let w be a weight. A functional a satisfies the D_r condition if there exists a finite constant C such that for each cube Q,

$$\sum_{P \in \Delta} a(P)^r\, w(P) \leqslant C^r a(Q)^r\, w(Q) \tag{48}$$

whenever Δ is a family of disjoint dyadic subcubes of Q. We denote the smallest constant C for which (48) holds by $\|a\|$.

In the Euclidean case, we consider a more general initial inequality which includes example (5). We need to introduce optimal polynomials of the following form. Fix a cube Q and a nonnegative integer m. The space $\mathcal{P}_m$ of real-valued polynomials of degree at most m is a Hilbert space with the inner product

$$\frac{1}{\mu(Q)} \int_Q fg\, d\mu.$$

Consider the orthonormal basis $\{\varphi_v\}$, $|v| \leqslant m$, obtained by applying the Gram–Schmidt orthonormalization process to the power functions $\{x^v\}$, $|v| \leqslant m$. Observe that

$$\|\varphi_v\|_{L^\infty(Q)} \leqslant C \left(\frac{1}{\mu(Q)} \int_Q |\varphi_v|^2\, d\mu \right)^{1/2} = C \tag{49}$$

since the space $\mathcal{P}_m$ is finite dimensional, and so all norms on it are equivalent. The constant C depends only on m. We let P_Q be the operator defined by

$$P_Q f(x) = \sum_{|v| \leqslant m} \frac{1}{\mu(Q)} \int_Q f\varphi_v\, d\mu\, \varphi_v(x),$$

which is a projection from $L^1(Q)$ onto $\mathcal{P}_m$. By (49) we have the following key property:

$$\|P_Q f\|_{L^\infty(Q)} \leqslant \frac{\gamma}{\mu(Q)} \int_Q |f(y)|\, d\mu(y). \tag{50}$$

Observe that when $m=0$, $P_Q f = \int_Q f\,d\mu/\mu(Q)$. These polynomials $P_Q f$ are optimal in the sense that

$$\inf_{\pi \in \mathscr{P}_m} \frac{1}{\mu(Q)} \int_Q |f-\pi|\,d\mu \approx \frac{1}{\mu(Q)} \int_Q |f-P_Q f|\,d\mu.$$

In fact we may replace the L^1 norm by any L^p norm, $1<p<\infty$. Indeed, the inequality in the direction "$\leqslant$" is trivial. To prove the opposite inequality, observe that since P_Q is a projection we have $P_Q \pi = \pi$ for any polynomial of degree at most m, and therefore

$$\begin{aligned}
&\frac{1}{\mu(Q)} \int_Q |f-P_Q f|\,d\mu \\
&\quad \leqslant \frac{1}{\mu(Q)} \int_Q (|f-\pi| + |P_Q(f-\pi)|)\,d\mu \\
&\quad \leqslant \frac{1}{\mu(Q)} \int_Q |f-\pi|\,d\mu + \|P_Q(f-\pi)\|_{L^\infty(Q)} \leqslant \frac{1+\gamma}{\mu(Q)} \int_Q |f-\pi|\,d\mu
\end{aligned}$$

by (50).

We have the following result, which for simplicity we state in the global form.

THEOREM 7.2. *Let $1 \leqslant r < \infty$ and suppose that a satisfies the weighted D_r condition* (48) *for some $w \in A_\infty(\mu)$. Let f be a locally integrable function such that for all cubes Q,*

$$\frac{1}{\mu(Q)} \int_Q |f-P_Q f|\,d\mu \leqslant \|f\|_a\, a(Q). \tag{51}$$

Then

(a) *There exists a constant c independent of f and a such that for all cubes Q,*

$$\|f-P_Q f\|_{L^{r,\infty}(Q,w)} \leqslant c\,\|a\|\,\|f\|_a\, a(Q). \tag{52}$$

(b) *Furthermore, if $r>1$ and p satisfies $1<p<r$, there exists a constant c independent of f and a such that for all cubes Q,*

$$\left(\frac{1}{w(Q)} \int_Q |f-P_Q f|^p\,w\,d\mu\right)^{1/p} \leqslant c\,\|a\|\,\|f\|_a\, a(Q). \tag{53}$$

Observe that the factor $a(Q)$ on the right sides of both (52) and (53) remains the same as in (51), i.e., there is no enlargement of Q; compare with Theorem 2.3.

We next state a corresponding result similar to Theorem 3.1. We assume that $b: \mathcal{Q} \times \mathcal{F} \to (0, \infty)$, with the same properties listed in Section 3, including (21) and the following version of (22): for some $r \geqslant 1$ and some $w \in A_\infty(\mu)$,

$$\sum_{P \in \Delta} b(P, f)^r\, w(P) \leqslant C^r b(Q, f)^r\, w(Q) \tag{54}$$

for all $f \in \mathcal{F}$, every cube Q and every family Δ of pairwise dyadic subcubes of Q.

THEOREM 7.3. *Let b be as above, m be a nonnegative integer, and suppose that the following initial condition holds for all $f \in \mathcal{F}$ and Q,*

$$\frac{1}{\mu(Q)} \int_Q |f - P_Q f|\, d\mu \leqslant c b(Q, f), \tag{55}$$

with c independent of f and Q. Then

$$\left(\frac{1}{w(Q)} \int_Q |f - P_Q f|^r\, w\, d\mu\right)^{1/r} \leqslant C b(Q, f) \tag{56}$$

with C independent of f and Q.

Let us briefly show how we recover the classical sharp $(L^{n'}, L^1)$ Poincaré inequality in $\mathbf{R}^n$. First, for Lipschitz continuous f, a well-known argument based on the Mean Value theorem yields

$$|f(x) - f_Q| \leqslant c \int_Q \frac{|\nabla f(y)|}{|x - y|^{n-1}}\, dy, \qquad x \in Q,$$

which when combined with Fubini's theorem and the growth estimate $\int_Q |x - y|^{1-n}\, dy \leqslant C\ell(Q)$, $x \in Q$, implies

$$\frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy \leqslant C \frac{\ell(Q)}{|Q|} \int_Q |\nabla f|\, dy. \tag{57}$$

This is the (L^1, L^1) Poincaré inequality. Now an application of Theorem 7.3 with $m=0$ produces the sharp version

$$\left(\frac{1}{|Q|}\int_Q |f-f_Q|^{n'}\,dy\right)^{1/n'} \leqslant C\,\frac{\ell(Q)}{|Q|}\int_Q |\nabla f|\,dy$$

since the functional on the right side of (57) satisfies the $D_{n'}$ condition.

We sketch below the proof of part (a) of Theorem 7.2. We skip the proofs of part (b) and of Theorem 7.3 since these would be repetitions of the corresponding proofs that we gave in the context of spaces of homogeneous type. To prove part (a), observe that we may assume that $\|f\|_a=1$. Fix a cube Q and for each $\lambda>0$, let $\Omega_\lambda=\{x\in Q: M(f-P_Q f)(x)>\lambda\}$ where $M=M_Q^d$ is the dyadic Hardy–Littlewood maximal function relative to Q with respect to μ, i.e.,

$$Mg(x)=\sup_{\substack{P\,:\,x\in P\\ P\in\mathscr{D}(Q)}}\frac{1}{\mu(P)}\int_P |g|\,d\mu,$$

where $\mathscr{D}(Q)$ denotes the family of all dyadic subcubes of Q. Then by the Lebesgue differentiation theorem we have $\{x\in Q: |f(x)-P_Q f|>\lambda\}\subset\Omega_\lambda$, except possibly for a set of μ-measure 0, and so (52) will follow from

$$\sup_{\lambda>0}\lambda^r\,\frac{w(\Omega_\lambda)}{w(Q)}\leqslant C^r\,\|a\|^r\,a(Q)^r. \tag{58}$$

Since $w\in A_\infty(\mu)$, there are constants $C,\delta>0$ such that $w(E)\leqslant C(\mu(E)/\mu(\tilde{Q}))^\delta\,w(\tilde{Q})$ for every cube $\tilde{Q}$ and every measurable set $E\subset\tilde{Q}$. We denote the smallest such constant C by $[w]_{A_\infty}$. As in the general situation, the key estimate is the following inequality of good-λ type:

Let f, Q, Ω_λ and w be as above. Then for all λ, ε, N with $\lambda>0$, $N>\gamma C_\mu$ and $0<\varepsilon\leqslant\|a\|$,

$$w(\Omega_{N\lambda})\leqslant\frac{\varepsilon^\delta[w]_{A_\infty}}{(N-\gamma C_\mu)^\delta}\,w(\Omega_\lambda)+\frac{\|a\|^r}{\lambda^r\varepsilon^r}\,a(Q)^r\,w(Q). \tag{59}$$

The estimate (58) follows from (59) by a standard good-λ argument.

To prove (59), observe first that the inequality is obvious if $\lambda\leqslant a(Q)$ since $\Omega_{N\lambda}\subset Q$ and $\varepsilon\leqslant\|a\|$. Therefore, we may assume that $\lambda>a(Q)$. Then, by (51),

$$\lambda>\frac{1}{\mu(Q)}\int_Q |f-P_Q f|\,d\mu.$$

We can consider the Calderón–Zygmund decomposition of $(f - P_Q f)\chi_Q$ relative to Q for these λ's. This yields a collection of dyadic subcubes of Q, $\{Q_i\}$, maximal with respect to inclusion, satisfying $\Omega_\lambda = \bigcup_i Q_i$ and

$$\lambda < \frac{1}{\mu(Q_i)} \int_{Q_i} |f - P_Q f| \, d\mu \leqslant C_\mu \lambda \tag{60}$$

for each integer i, where C_μ is the doubling constant of μ. Now let $N > C > 1$, where $C = \gamma C_\mu$. Since $\Omega_{N\lambda} \subset \Omega_\lambda$, we have

$$w(\Omega_{N\lambda}) = w(\Omega_{N\lambda} \cap \Omega_\lambda) = \sum_i w(\{x \in Q_i : M((f - P_Q f)\chi_Q)(x) > N\lambda\}) =$$

$$= \sum_i w(\{x \in Q_i : M((f - P_Q f)\chi_{Q_i})(x) > N\lambda\})$$

by the maximality of each of the cubes Q_i. Now let $E_{Q_i} = \{x \in Q_i : M((f - P_{Q_i} f)\chi_{Q_i})(x) > (N - C)\lambda\}$. Then, for $x \in Q_i$, since P_Q leaves polynomials of degree at most m fixed, we have

$$\begin{aligned} |f(x) - P_Q f(x)| &\leqslant |f(x) - P_{Q_i} f(x)| + |P_{Q_i}(f - P_Q f)(x)| \\ &\leqslant |f(x) - P_{Q_i} f(x)| + \|P_{Q_i}(f - P_Q f)\|_{L^\infty(Q_i)} \\ &\leqslant |f(x) - P_{Q_i} f(x)| + \frac{\gamma}{\mu(Q_i)} \int_{Q_i} |f - P_Q f| \, d\mu \\ &\leqslant |f(x) - P_{Q_i} f(x)| + C\lambda \end{aligned}$$

by (50) and (60). Thus, if $N > C$,

$$w(\Omega_{N\lambda}) \leqslant \sum_i w(E_{Q_i}) \leqslant \sum_{i \,:\, a(Q_i) < \varepsilon\lambda} w(E_{Q_i}) + \sum_{i \,:\, a(Q_i) \geqslant \varepsilon\lambda} w(E_{Q_i}).$$

To estimate the first sum we use the A_∞ condition on the weight w together with the weak type $(1, 1)$ property (with constant one) of M^d; the second sum uses the D_r condition:

$$\begin{aligned} w(\Omega_{N\lambda}) &\leqslant [w]_{A_\infty} \sum_{i \,:\, a(Q_i) < \varepsilon\lambda} \left(\frac{\mu(E_{Q_i})}{\mu(Q_i)} \right)^\delta w(Q_i) + \sum_{i \,:\, a(Q_i) \geqslant \varepsilon\lambda} \left(\frac{a(Q_i)}{\varepsilon\lambda} \right)^r w(Q_i) \\ &\leqslant [w]_{A_\infty} \sum_{i \,:\, a(Q_i) < \varepsilon\lambda} \left(\frac{1}{(N - C)\lambda} \frac{1}{\mu(Q_i)} \int_{Q_i} |f - P_{Q_i} f| \, d\mu \right)^\delta w(Q_i) \\ &\quad + \frac{1}{(\varepsilon\lambda)^r} \sum_i a(Q_i)^r \, w(Q_i) \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{[w]_{A_\infty}}{(N-C)^\delta} \sum_{i\,:\,a(Q_i)<\varepsilon\lambda} \left(\frac{a(Q_i)}{\lambda}\right)^\delta w(Q_i) + \frac{\|a\|^r}{(\varepsilon\lambda)^r} a(Q)^r\, w(Q) \\
&\leqslant \frac{[w]_{A_\infty}}{(N-C)^\delta}\ \varepsilon^\delta \sum_i w(Q_i) + \frac{\|a\|^r}{(\varepsilon\lambda)^r}\, a(Q)^r\, w(Q) \\
&\leqslant \frac{\varepsilon^\delta [w]_{A_\infty}}{(N-C)^\delta}\, w(\Omega_\lambda) + \frac{\|a\|^r}{(\varepsilon\lambda)^r}\, a(Q)^r\, w(Q).
\end{aligned}
$$

This proves the desired good-λ inequality (59) and also completes the proof of part (a) of Theorem 7.2. ▮

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REFERENCES

[BCLSC] D. Bakry, T. Coulhon, M. Ledoux, and L. Saloff-Coste, Sobolev inequalities in disguise, *Indiana Univ. Math. J.* **44** (1995), 1033–1074.

[BM1] M. Biroli and U. Mosco, Sobolev inequalities on homogeneous spaces, *Potential Anal.* **4** (1995), 311–324.

[BM2] M. Biroli and U. Mosco, A Saint-Venant type principle for Dirichlet forms on discontinuous media, *Ann. Mat. Pura Appl. (4)* **169** (1995), 125–181.

[BG] D. L. Burkholder and R. F. Gundy, Extrapolation and interpolation of quasilinear operators on martingales, *Acta Math.* **124** (1970), 249–304.

[BS] C. Bennett and R. C. Sharpley, "Interpolation of Operators," Pure Appl. Math., Vol. 129, Academic Press, Boston, 1988.

[Bo] B. Bojarski, "Remarks on Sobolev Imbedding Inequalities" Lecture Notes in Math., Vol. 1351, pp. 52–68, Springer-Verlag, Berlin/New York, 1989.

[Bu] P. Buser, A note on the isoperimetric constant, *Ann. Sci. Ecole Norm. Sup.* **15** (1982), 213–230.

[CDG] L. Capogna, D. Danielli, and N. Garofalo, Subelliptic mollifiers and a basic pointwise estimate of Poincaré type, *Math. Zeit.* **226** (1997), 147–154.

[C] S.-K. Chua, Weighted Sobolev inequalities on domains satisfying the chain condition, *Proc. Amer. Math. Soc.* **117** (1993), 449–457.

[CW] R. Coifman and G. Weiss, "Analyse harmonique non-commutative sur certains espaces homogènes," Lecture Notes in Math., Vol. 242, Springer-Verlag, Berlin/New York, 1971.

[ChW] S. Chanillo and R. L. Wheeden, Weighted Poincaré and Sobolev inequalities and estimates for weighted Peano maximal functions, *Amer. J. Math.* **107** (1985), 1191–1226.

[DS] R. A. DeVore and R. S. Sharpley, Maximal Functions Measuring Smoothness, *Mem. Amer. Math. Soc.* **293** (1984).

[EG] L. C. Evans and R. Gariepy, “Measure Theory and Fine Properties of Functions,” Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[FP] C. Fefferman and D. H. Phong, Subelliptic eigenvalue estimates, *in* “Conference on Harmonic Analysis, Chicago” (W. Beckner *et al.*, Eds.), pp. 590–606, Wadsworth, Belmont, CA, 1981.

[FLW] B. Franchi, G. Lu, and R. L. Wheeden, Representation formulas and weighted Poincaré inequalities for Hörmander vector fields, *Ann. Inst. Fourier* **45** (1995), 577–604.

[FGaW] B. Franchi, S. Gallot, and R. L. Wheeden, Sobolev and isoperimetric inequalities for degenerate metrics, *Math. Ann.* **300** (1994), 557–571.

[FGuW] B. Franchi, C. E. Gutiérrez, and R. L. Wheeden, Weighted Sobolev–Poincaré inequalities for Grusin type operators, *Comm. Partial Differential Equations* **19** (1994), 523–604.

[FW] B. Franchi and R. L. Wheeden, Some remarks about Poincaré type inequalities and representation formulas in metric spaces of homogeneous type, *J. of Inequalities and Applications*, to appear.

[GHL] S. Gallot, D. Hulin, and J. Lafontaine, “Riemannian Geometry,” 2nd ed., Springer-Verlag, Berlin/New York, 1990.

[GN] N. Garofalo and D.-M. Nhieu, Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces, *Comm. Pure Appl. Math.* **49** (1996), 1081–1144.

[HaK] P. Hajlasz and P. Koskela, Sobolev meets Poincaré, *C. R. Acad. Sci. Paris* **320** (1995), 1211–1215.

[HKM] J. Heinonen, T. Kilpelainen, and O. Martio, “Nonlinear Potential Theory of Degenerate Elliptic Equations,” Oxford Univ. Press, Oxford, 1993.

[HK1] J. Heinonen and P. Koskela, From local to global in quasiconformal structures, *Proc. Nat. Acad. Sci. USA* **93** (1996), 554–556.

[HK2] J. Heinonen and P. Koskela, Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.*, to appear.

[Je] D. Jerison, The Poincaré inequality for vector fields satisfying Hömander’s condition, *Duke Math. J.* **53** (1986), 503–523.

[Jo] J. L. Journé, “Calderón–Zygmund Operators, Pseudo-Differential Operators and the Cauchy Integral of Calderón,” Lecture Notes in Math., Vol. 994, Springer-Verlag, Berlin/New York, 1983.

[LN] R. Long and F. Nie, “Weighted Sobolev Inequality and Eigenvalue Estimates of Schrödinger Operators,” Lecture Notes in Math., Vol. 1494, pp. 131–141, Springer-Verlag, Berlin/New York, 1990.

[L] G. Lu, The sharp Poincaré inequality for free vector fields: An endpoint result, *Rev. Mat. Iberoamericana* **10** (1994), 453–466.

[M] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* **165** (1972), 207–226.

[MP] P. MacManus and C. Pérez, Generalized Poincaré inequalities: Sharp self-improving properties, *Intern. Math. Res. Notices*, to appear.

[MS] R. Macias and C. Segovia, Lipschitz functions on spaces of homogeneous type, *Adv. Math.* **33** (1979), 257–270.

[MSC] P. Maheux and L. Saloff-Coste, Analyse sur les boules d’un opérateur sous-elliptique, *Math. Ann.* **303** (1995), 713–740.

[NSW] A. Nagel, E. M. Stein, and S. Wainger, Balls and metrics defined by vector fields, I: Basic properties, *Acta Math.* **155** (1985), 103–147.

[SCo1] L. Saloff-Coste, A note on Poincaré, Sobolev and Harnack inequalities, *Internat. Math. Res. Notices* **2** (1992), 27–38.

[SCo2] L. Saloff-Coste, On global Sobolev inequalities, *Forum Math.* **6** (1994), 271–286.

[SCo3] L. Saloff-Coste, Uniformly elliptic operators on Riemannian manifolds, *J. Differential Geom.* **36** (1992), 417–450.

[SCa] A. Sánchez-Calle, Fundamental solutions and geometry of the sums of squares of vector fields, *Invent. Math.* **78** (1984), 142–160.

[SW] E. T. Sawyer and R. L. Wheeden, Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces, *Amer. J. Math.* **114** (1992), 813–874.

[W] R. L. Wheeden, A characterization of some weighted norm inequalities for the fractional maximal function, *Studia Math.* **107** (1993), 251–272.